Niveau(x) d'enseignement: 0 Numéro EDUQ : 9570178
Centre d'intérêt: 841

Borasi, Raffaella.

A pedagogical use of errors for students' construction of mathematical knowledge / par Raffaella Borasi. -- Montréal : Université du Québec à Montréal, Centre interdisciplinaire de recherche sur l'apprentissage et le développement en éducation, 1989.
33 f. : ill. ; 22 cm. -- (Séminaire sur la représentation ; 39)

Bibliogr.: f. 32-33.

Vedette(s)-secondaire(s): Université du Québec à Montréal. Centre interdisciplinaire de recherche sur l'apprentissage et le développement en éducation

Descripteur(s): Mathématiques * Expérimentation pédagogique

Identificateur(s) de sujet: Erreur

Dans ce document, l'auteure propose d'utiliser les erreurs comme moyen de motivation et d'exploration des mathématiques. Elle fait part, d'abord, des positions des auteurs de différentes disciplines sur cette question. En second lieu, elle démontre, à l'aide d'un exemple, l'utilisation pédagogique des erreurs comme instrument d'investigation mathématique. Finalement, elle rapporte les résultats d'une expérimentation pédagogique en ce sens.

CENTRE INTERDISCIPLINAIRE DE RECHERCHE SUR
L'APPRENTISSAGE ET LE DÉVELOPPEMENT EN ÉDUCATION

ciradé

Université du Québec à Montréal

15-70178

A Pedagogical use of errors for students' construction of mathematical knowledge

par Raffaella Borasi
University of Rochester, USA

Séminaire sur la Représentation
10 mars 1989 - nº 39

Note: La publication de ce séminaire est rendue possible grâce à une subvention du fonds FCAR.

I- Introduction

While educators have been concerned with students errors for a long time, most of that history documents efforts by teachers as well as researchers to use those errors as tools for diagnosing learning difficulties and finding strategies for eradicating them. The history and the philosophy of mathematics, on the contrary, make abundantly clear the way the discipline evolved as a result of creative uses of supposed errors, regardless of their elimination. Could students, too, benefit from mathematical errors in a similar way?

In this paper, I suggest that indeed errors can constitute the motivation and means to explore specific mathematical topics and to inquire into the nature of mathematics, and thus provide students with a valuable heuristic in their construction of mathematical knowledge, without at the same time necessarily requiring considerable mathematical background or the mind of a genius. This alternative interpretation of the role of errors as *springboards for inquiry* will be presented and analyzed as follows.

First of all, I will analyze different views on the roles and uses of errors expressed by authors in a variety of fields. This brief review will suggest new ways in which errors could be used constructively in mathematics instruction. The pedagogical interpretation of errors as *springboards for inquiry* will then be identified and illustrated by means of a brief example. The results from a teaching experiment conducted with two 16-year-old students will then be reported, and used to contribute empirical data for the discussion of the educational potential of the strategy of using errors as *springboards*. The paper will conclude with a brief discussion on the implication of the suggested strategy for mathematics instruction.

II- Views about Errors across Fields

Many authors in different areas of philosophy, education and science have considered the role of errors, more or less explicitly. While it will not be possible within the constraints of this paper to exhaustively review all the existing literature on the topic, this section will briefly report major contributions that allow to identify alternative approaches to errors with implications for the use of errors in mathematics education.

"Error Analysis" in Mathematics Education

Since the beginning of the century there has been interest and research on errors in mathematics education (see for example Radatz [1979, 1980], Norak-Helm [1983], and CIEAEM [1988], for reviews and collections of papers to this regard). Despite the great number of works in the area, it seems possible to recognize in all these studies a few common features and assumptions that I will try to summarize here.

First of all, these studies focus on *student* errors in mathematics. They challenge the behavioristic explanation of errors as due to "unsureness and carelessness," and set themselves the task of proving that students errors:

- are causally determined, and very often systematic;
- are persistent and will last for several school years, unless the teacher intervenes pedagogically;
- can be analyzed and described as error techniques;
- can be derived, as to their causes, from certain difficulties experienced by students while receiving and processing information in the mathematical learning process, or from effects of the interaction of variables acting on mathematics education (teacher, curriculum, student, academic environment, etc.).

The overall interpretation of mathematical errors is here closely connected with a concern for remediation, as it is well illustrated again by the following quote by Radatz:

> Student errors "illustrate" individual difficulties; they show that the student has failed to understand or grasp certain concepts, techniques, problems, etc., in a "scientific" or "adult" manner. Analyzing a student's errors may reveal the faulty problem-solving process and provide information on the understanding of and the attitudes towards mathematical problems. (Radatz, 1980, p. 16).

Given these overall beliefs about errors, it is not surprising that "error analysis" research in mathematics education, especially in the U.S., mainly focussed on:

- attempting to determine and classify potential causes of errors;
- identifying potential error techniques (especially in areas of the curriculum such as arithmetic);
- determining the frequency distribution of these errors techniques;
- attempting to classify and group errors;
- determining the persistency of individual error techniques;

- (at a lesser degree) developing didactical aids for treating particular learning difficulties and errors.

The results of these studies have undoubtedly provided interesting information with respect to both research on the learning process in mathematics, and classroom practice.

As pointed out in Radatz' article, this kind of error analysis recognized the educational potential of errors as "an opportunity to diagnose learning difficulties" (Radatz, 1980, p. 16). It also recognized, but used at a lesser degree, other aspects of the role of errors in mathematics education, such as, the fact that student errors can provide "the starting point for research on the mathematics teaching-learning process" (Radatz, 1980, p. 16).

Different perspectives on errors can be gathered looking at research in other fields and responding to different concerns, as I hope to be able to show in the following sections.

Uses of Errors in Some Psychological Studies

A deeper realization of the potential of errors as tools to see into how a mind works can be found in several psychological studies, of which Piaget's work can be considered representative. For example, in Piaget's identification of several stages in the cognitive development of a child, a major role was played by the study of key "errors" such as the lack of specific conservation principles until a certain age. The presence and successive disappearance of these and other "errors" permitted in fact to mark the difference between certain points of the development, and more importantly to characterize by comparison the stages themselves. It is interesting to notice that in Piaget's work these "errors" are seen as a necessary step in the process. Consequently, no concern is raised about their elimination.

On the contrary, in applications of Piaget's theories to education and other more didactically oriented studies on the learning process in mathematics, a concern for remediation is often present. Errors are still used to inquire into children's conceptions and learning process. But this is done with the goal in mind of helping teachers improve mathematics curricula and teaching methodologies in order to prevent, or at least minimize the future occurrence of these errors in their students. In mathematics education, interesting examples of research which uses errors, more or less explicitly, in

Some authors such as Papert (see Papert [year]) have recognized and exploited in a more radical way the educational potential of [illegible] working with computers.

this way are Fischbein's studies on children's conceptions of infinity (Fischbein et al., 1979, 1981), and Vergnaud's analysis of conceptual schemes for subtraction (Vergnaud, 1982).

Debugging Process in Computer Science

So far we have always seen the teacher or the researcher involved in the rewarding analysis of other people's errors. And indeed that is what we are used to expect in almost any educational setting or subject matter: student errors are recognized, pointed out, explained and eventually remediated by an authority.

However, this familiar pattern fails as soon as the student is involved in some computer programming. In that context it is easy enough to recognize the presence of some error. The program does not work as its creator wished, or may be it does not run at all! Now, however, the student himself/herself is expected to identify and eliminate the error, with minimal help from the outside (in the form of computer error messages, or the appeal to a computer consultant). As a computer science student, one has to learn to "debug" one's own programs, i.e. develop some strategies to diagnose errors and suitably remediate them.

Though often a difficult task, the debugging process can turn out to be a challenging problem solving activity, and can often bring along a deeper understanding of the problem in question as well as a better knowledge of the potential and limitations of the computer language used.

The analogy of "error analysis" to this "debugging process" has been recognized in mathematics education, as shown by the recent use of the jargon "bug" and "debug" in this area, too. However, most educators seem not to have caught the real novelty and challenge as they move from the practice of debugging in computer science to traditional error analysis. That is, the role of "debugger" should pass from the teacher to the students themselves if we want to exploit the power of errors as real motivation to problem solving and the analysis of errors as a way to improve understanding in any subject matter.

Some authors such as Papert (see Papert, 1980), however, have recognized and exploited in a more radical way the educational potential of errors for exploration when working with computers.

Errors in the History of Science

When we consider student mistakes, we are always comparing them with an already established body of knowledge. It is easier, therefore, to view errors as shortcomings of the learning process and their elimination as an educational objective.

The consideration of the errors which occurred among scientists in the history of science can provide a different and interesting perspective. What is the role of errors in the creation of new knowledge?

The work of philosophers and historians of science such as Kuhn (1970), Lakatos (1976) and M. Kline (1980) can help us realize that errors play a fundamental role in the growth of a discipline. The history of mathematics, for example, has shown on more than one occasion that it is possible to capitalize on errors in various ways, regardless of any tendency to repeat or eliminate them. Sometimes, failures to meet an initial goal has led to unexpected and revolutionary results. Think for example of Sacchieri's creation of a non-Euclidean geometry, which was generated by an unsuccessful attempt to "prove" the parallel postulate. The problems and contradictions encountered in the early development of the calculus also provide a very interesting example of the role of errors in the history of mathematics. In fact, the errors made initially in this field shook the mathematicians confidence to the point of motivating a revision of the methodology used in the discipline. The search for rigor eventually uncovered profound questions about the foundations of mathematics.

Less spectacular, but equally important for the growth of mathematical knowledge, is the use that mathematicians make of their own errors in their everyday research. Incorrect conjectures, unjustified guesses, and partial results are all necessary and invaluable steps in the creation of new mathematical results. Lakatos (1976) has provided very interesting historical examples of this aspect of mathematical discovery.

Kuhn's interpretation of the development of the history of science is also especially helpful to understand the special role of errors in the creation of new knowledge. Kuhn (1970) suggests that science develops along two complementary modes, which alternate to each other in time. Most of the time, we have what he calls "normal science". This is the development of research within an accepted "paradigm", which we can roughly describe as a general frame of reference, collecting a specific set of theoretical and methodological assumptions about the subject in question. In studying and developing the consequences of a paradigm, however, scientists inevitably find some unacceptable results or unsolvable problems that challenge the validity of the

paradigm itself. The search for an alternative paradigm thus motivated gives rise to a period of "extraordinary science" or "scientific revolution". In Kuhn's words:

> Confronted with anomaly or crisis, scientists take a different attitude towards existing paradigms, and the nature of their research changes accordingly. The proliferation of competing articulations, the willingness to try anything, the expression of explicit discontent, the recourse to philosophy and to debate over fundamentals, all these are symptoms of a transition from normal to extraordinary research... Scientific revolutions are inaugurated by a growing sense... that an existing paradigm has ceased to function adequately in the exploration of an aspect of nature to which that paradigm inself had previously led the way (Kuhn, 1970, pp. 90-91).

Some of the examples of "anomalies" reported by Kuhn have probably been seen by scientists of the time, or are now seen by looking in retrospective, as some sort of "error". An interpretation of at least certain errors as "anomalies a la Kuhn" provides a new perspective also about the role of errors for learning. It suggests in fact their employment as motivation and strarting point for reflection on specific theories, and to eventually foster conceptual change.

These views about the development of science suggest that an analysis of errors is a worthwhile activity for scientists, and it may lead to results other than the elimination of the error as initially expected. It seems worth questioning, then, whether it is possible to engage students in a somewhat similar activity of "research" around an error, and what consequences for the learning and understanding of a discipline such an activity could have.

III- Errors as "Springboard for Inquiry" in Mathematics Instruction

The analysis of the existing literature conducted in the previous section, however sketchy, has provided ground for a better evaluation of the uses of errors currently recognized in mathematics instruction.

On one hand, the "error analysis" approach which most mathematics educators seem to share represents a radical step forwards in an appreciation of the positive role of errors for mathematics instruction. Rather than seeing student errors as merely an incorrect response to be eradicated - as suggested by a behavioristic perspective - teachers and researchers have learned to take advantage of the information those errors can provide. Through the analysis of student errors, mathematics education has gained new insight about some of the learning difficulties many students experience as well as about the mathematical learning process. This in turn has suggested the creation of more

(ii) A set of points with a radius

effective teaching approaches to either remediate the existing errors and misconceptions, or present the same content in such a way that similar errors could be avoided by future students.

Without playing down the important contributions that an "error analysis" approach has thus brought to mathematics education, I would like to argue that it has also only partially exploited the educational potential of errors.

First of all, within this approach only teachers and researchers are engaged in the creative act of analyzing an error. The educational values recognized in the practice of *debugging* in the context of computer programming, on the contrary, suggest that the students themselves could benefit from an ctive analysis of their own errors.

Secondly, in an "error analysis approach" the goal of the activity is essentially to diagnose the causes of the errors so as to eliminate them more efficiently. In other words, errors are still seen as a signal that something has gone wrong in the learning process and remediation is needed. An analysis of how errors are used in other contexts such as computer science, scientific research and cognitive psychology, would suggest on the contrary that there are other viable ways to capitalize on errors, regardless of any tendency to repeat or eliminate them. As complementary to an "error analysis" approach, I thus suggest that mathematics educators recognize and make use of the potential of errors to invite students to raise and pursue new questions and inquires in mathematics.

To briefly illustrate the variety of directions in which such inquiries could develop, as well as the possibility of using errors as *springboards* without any great mathematical knowledge, sophistication or ability[1], I will now present an application of the strategy on the following definitions of circle, produced by some mathematics students:

(1) Locus of points equidistant from a given point.
(2) A closed, continuous, rounded line.
(3) A straight line which changes directions constantly.
(4) Something whose area is πR^2.
(5) A set of points with a radius. Round thing.

---

1 Other and better developed, examples of instructional activities developed around the in-depth analysis of a mathematical error can be found in Borasi [1986(a), 1986(b), 1987] and Brown and Callahan [1985].

No doubt all of the above definitions are incorrect, at least from the point of view of a mathematician. However, they also contain interesting intuitions about the nature and properties of circles, and reflect different interpretations of what a definition should be and do in mathematics. Therefore, I would like to argue that the analysis of such incorrect definitions could help students gain a deeper understanding of notions such as *circle, Euclidean geometry or mathematical definition.*

First of all, the above definitions could be analyzed along the following directions:

- *What mathematial properties can be derived from each of these definitions? Which ones fit our image of 'circle'? Which ones don't?*
- *What other mathematical objects could be described by some of these definitions?*
- *In each definition, are all the properties stated really essential? Could any be eliminated?*
- *How could we "fix up" these definitions?*

As a result of these activities, we could discover some unusual properties of circles, realize the possibility of defining *circle* in alternative yet correct ways, and better distinguish circles from closely related figures such as spheres, ellipses or straight lines.

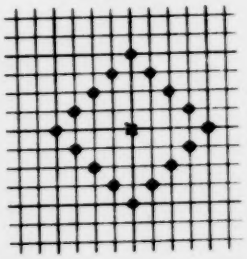
Because they present a contrat with the standard definition of *circle*, some of these incorrect definitions can also lead to more open-ended explorations, once the standard characterization of circles is challenged through questions such as:

- *What if this were the correct definition for the concept? What would be the concept itself?*
- *What would be the consequences of accepting this definition in mathematics?*
- *How could this definition be further modified? What alternative notions could then be created?*
- *Why did mathematicians choose the standard definition?*

Thus students could, for example, be led to explore alternative notions of *circle* in the context of topology and projective geometry, or even in non-standard areas of mathematics such as a son-Euclidean geometry. In fact, dfn.(1) will give rise to geometric figures with different shapes and properties, depending upon the specific distance considered and the constraints of the context in which we are operating. If we are in a city, for instance, the points which can be reached by driving exactly one mile from a given point are probably coming closer to from a square than a usual circle - since the car has to follow the streets and cannot go across buildings! In figure 1, the reader

can see what a "circle" of radius 4 units looks like in *taxi-geometry* (Papy and Papy, 1973; Borasi, 1981; Krause, 1986), where distance is defined as "the length of the shortest path connecting two points on a square-grid" - the square-grid representing an idealization of the regular street pattern of a modern city like Manhattan.

FIGURE 1



It seems the worthwile to explore what properties of *circles* are yet maintained in a new situation such as *taxi-geometry*, and in turn whether and how the notion of *circle* itself could be generalized to <u>any</u> metric space.

An investigation into the notion of *mathematical definition* can also be stimulated and conducted through the analysis of the above definitions with respect to the *kind of error* the contain. Such inquiry could be led by questions such as:

- *What precisely constitutes the "error" in each of these definitions?*
- *Are there different degrees of "wrongness" in a mathematical definition?*
- *What do these definitions try to accomplish? Could a certain definition be considered acceptable in certain contexts but not in others?*

These questions can lead us to discuss the basic properties of mathematical definitions, how such definitions differ from definitions in other fields, and to realize that the acceptability of a given definition may also depend on the context. It may in fact be easier to identify what causes problems in specific incorrect definitions - such as the use of vague terminology in dfn.(5) or the inclusion of unwanted instances in dfn.(2) - and from there abstract some more general criteria which mathematical definitions should satisfy. At the same time, the realization that the vagueness built in dfn.(1) may prove a springboard for generalizations of the notion of *circle* to metric spaces other than the Euclidean plane, could help us appreciate that even those criteria are not absolute and

that ambiguity can have some value in mathematics too.

This example well illustrates the variety of directions for exploration that the in-depth analysis of some mathematical error could stimulate. Notice how the inquiry initiated did not seem to require considerable technical background, and thus would be accessible to most secondary mathematics students. At the same time, it could involve students in valuable problem solving and problem posing activities, and even on reflections about the nature of mathematics as a discipline - crucial components in the mathematical education of our students, and yet mostly neglected in the current pre-college mathematics curricula.

In the following section, I will try to provide empirical evidence that secondary students can indeed engage productively in a use of mathematical errors as *springboards for inquiry*. Such data will also provide the basis to suggest potential benefits and implications of this strategy for mathematics education.

IV- Results from a Teaching Experiment

Overview

To allow for a full exploitation of the strategy of using errors as springboards for inquiry and an in-depth analysis of its outcomes, its first implementation was conducted within the context of a *teaching experiment*. A unit on mathematical *definitions* was designed making use whenever appropriate of specific errors to motivate and engage the students in an inquiry into this complex mathematical notion, with the expectation that in addition the teacher would attempt to take advantage as much as possible of any unexpected error that the students might make in the course of instruction.

An ideal occasion to implement this unit presented itself when two students, who for various reasons had missed many classes in a course I had previously taught, asked for an opportunity to "make up" for these absences and thus be still allowed to get credit for the course.

To correctly interpret the results of the experience reported in the following pages, it will be important to realize that the subjects (who will be called Kim an Mayl were far from being enthusiastic "mathematically gifted students". Though both quite bright girls, neither of these students was espacially interested in mathematics, and they had

not always had positive experiences with their mathematics courses in the past. Since they were both experiencing serious personal problems at the time the teaching experiment was conducted, I had also to deal with several absences and uneven commitments.

The teaching experiment involved ten instructional sessions of about 40 minutes each, in which both students always participated. The sessions were audio-taped and transcribed, and detailed narrative reports of the same prepared, integrating the dialogue from the transcripts with other data and observations provided by the field notes of a non-participant observer. In addition, data to evaluate the effects of the experience were collected through various interviews with the students and a final project.[2]

The provide the reader, with an idea of the kind of activities the two students engaged in when using errors as *springboards*, their reactions to them and the benefits they derived from them, within the constraints of this paper, I have decided to report in-depth on three significant episodes which occurred in the teaching experiment. They will be proceeded by a brief sketch of the design of the whole unit, so as to allow the reader to better understand the context in which each episode took place. Finally, I will try to summarize how this experience can contribute to a better understandig of the strategy of using errors as *springboards for inquiry*.

Overall Design of the Instructional Unit and Use of Errors

Since mathematics students encounter definitions very early in the mathematics curriculum, it may be assumed that they know how to deal with mathematical definitions. However, this is not generally the case. High school students seem rarely to have a clear idea of the role of mathematical definitions, the difference between definition and properties of a mathematical concept, or the criteria mathematicians require in mathematical definitions. One major goal of this unit was thus to provide experiences that would allow the students to acquire these notions, so that they would be able to use definitions better in their future mathematics courses.

---

2 I refer the reader interested in further information on this experience to the report prepared for the National Science Foundation (Borasi, 1988), where the detailed narrative and commentary of each session, as well as for a more thorough analysis of the whole experience, can be found.

In addition, I wanted to make the students aware of the limitations, as well as the power, of mathematical definitions. More specifically, I wanted students to realize that even mathematical definitions cannot always be totally "precise" (since their interpretation may in part depend upon the context in which they are introduced), that there may be "exceptions" to a mathematical definition to be considered, and that a rigorous definition may not be possible or suitable for all mathematical objects. I hoped that these considerations would make the students aware of the similarities that definitions in mathematics have with definitions in other "more humanistic" fields, and invited them to reflect on their views of mathematics.

Discussing the nature of mathematical definitions in abstracts is an arduous task, even more so for secondary school students. To make this possible, the first step in this direction was the creation of a series of activities, mostly organized around the in-depth analysis of specific incorrect definitions, which I hoped would stimulate the students to question and explore the nature of definitions and the criteria for their evaluation in mathematics.

I chose to focus on the concepts of 'circle', 'polygon', 'equation' and 'exponent', because they all dealt with fundamental yet elementary mathematical notions. Students in their last years of high school could be expected to have some degree of familiarity with these concepts, and a sufficient technical background to work with them. At the same time, from the point of view of mathematical definitions these concepts present quite different characteristics.

'Circle' is a very familiar figure, which students can easily identify, and which can be described mathematically by a precise and simple definition in more than one way. I planned first to take advantage of these elements by asking students to evaluate a list of incorrect definitions of 'circle', with the goal of having them recognize the need for elements such as the use of precise terminology, isolation of the concept, and so on. At the same time, by means of this exercise as well as the application of alternative definitions of 'circle' in the derivation of some mathematical results, I hoped to make them aware of the possibility and value of having more than one way of defining the same mathematical concept. An interpretation of the common definition of 'circle' in the context of taxigeometry would then show the importance of considering the context in which one is operating, in order to interpret or evaluate a mathematical definition.

The concept of 'polygon' is also elementary, yet much less familiar than 'circle' to high school students. I expected that, if I did not provide them with a definition of 'polygon' to start with, the students would only have a vague idea 'a priori' of what figures

should or should not be considered instances of this concept. Thus the activity of creating a reasonable definition of 'polygon' for themselves would provide a context in which they might experience the situation of a mathematician who wants to define a new notion. After having the students grope for a while with tentative definitions, I could then provide some direction to their activity by suggesting the consideration of a tentative theorem: "The sum of the interio angles of an n-sided polygon measures $(n - 2) \times 180^{0}$" (a la Lakatos [1976]).

Students are usually familiar with the notion of 'exponent' as "the number of times you have to multiply a number by itself". However, they may not always realize that this definition applies only to exponents which are whole numbers. And yet, the notion of exponent can be extended to other numbers (such as negative integers or fractions) *at a cost*. This cost may include relinquishing the original intuitive meaning of the operation and dealing with some "exceptions" - such as 0^{0}. An analysis of the dangers, as well as the advantages, of these extensions of the operation of exponentiation could bring new elements into the students' consideration of the nature of mathematical definitions, by illustrating the intrinsic limitations which mathematicians may sometimes have to deal with.

I expected that the problems experienced while working with the definitions of 'polygon', 'exponent', as well as 'circle' in taxigeometry, would challenge some of the "absolutist" expectations that the students might have developed in the first activities focusing on the more familiar notion of 'circle'. In addition, I thought it would be worthwile to direct the students' attention to notions which could be intuitively understood and used by them, yet be so complex that a precise definition would be out of their reach. I planned to use the concept of 'variable' for this purpose.

The use of *errors as springboard* occurred continuously throughout the sessions, in two major modes:

1. Errors (previously selected by the researcher) were used to create experiences which would provide the appropriate context for the discoveries/discussions about the nature of mathematical definitions which I wanted to provoke the students with (for example, the analysis of wrong definitions of *circle* was designed to lead to the realization of the criteria usually employed in mathematics to evaluate definitions; some errors caused by overgeneralization were introduced to show some limitations that occur when I extend the definition of *exponent* to other numbers besides positive integers);

2. Some errors, made by the students during the process of discovery itself, were exploited impromptu, leading either to a constructive contribution in the process of discovery/solution itself, or in other cases to an exploration/digression that the researcher had not previously planned.

In-Depth Analysis of Significant Episodes

To illustrate the variety of activities and discussions experienced by the students involved in the teaching experiment, as well as the quality of their achievements, I have selected three episodes presenting some interesting differences in terms of mathematical context, aspects of definitions considered, and timing within the unit.

In the first episode, I will report the variety of questions and mathematical activities which were generated by the students' analysis of a list of incorrect definitions of 'circle'. This experience occurred at the very beginning of the unit, and took most of the 40 minutes allotted for our second session.

The second episode occurred towards the middle of the experience, and dealt with the students' attempts to define 'polygon' - a notion they did not feel very familiar with. This activity developed through our fourth and fifth meeting, taking about 45 minutes altogether.

Finally, I will report on the considerable challenges presented to the students by the examination of the usual definition of 'circle' in the context of taxigeometry. About half an hour of our eighth session was devoted to this topic.

All the dialogues reported in this as well as the following section are excerpts taken verbatim from the transcripts of instructional sessions and/or interviews. The following abbreviations will be employed: K. = Kim; M. = May; R. = researcher/instructor.

Episode 1: Analyzing Definitions of 'Circle'

Our starting point in this activity consisted in the following list of definitions of 'circle', previously selected amongst those produced by the two students and by other students of similar mathematical background.

A. All the possible series of points equidistant from a single point (May)

B. Circumference πr^2, = radius, an exact center, 360º (Kim)

C. *Round - 3.14 - shape of an orange, coin, earth - Pi.*

D. *Circle = something whose area is = to* πr^2.

E. *Circle:* $x^2 + y^2 = r^2$. *Round.*

F. *A circle is a geometric figure that lies in a two dimensional plane. It contains 360º and there is a point called the center that lies precisely in the middle. A line passing through the center is called the diameter. 1/2 of the diameter is the radius. I don't like circles too much any more because they look like big fat zeros but they can be fun because you can make cute little smiley faces with mohawks out of them.*

G. *A closed, continuous, rounded line.*

H. *I sometimes find myself going around in them...*

When asked to comment on these definitions, May spontaneously started to criticize a few of them on the ground that they were describing also figures other than circles:

M.: "Close continuous rounded line": that could just be a spiral... a closed one (she draws the following figure)

FIGURE 2

Definition H was also immediately rejected by both students because it was "too vague". While both students seemed satisfied with May's definition (def. A), their belief was challenged by the instructor's observation that a ball would also satisfy that description. May, however, was quick in recognizing where her original definition was falling short, and quickly produced the following modification:

All the possible series of points <u>on a plane</u> equidistant from a single point.

The consideration of def. D, followed by Kim's own dissatisfaction with her definition (def. B), brought up more explicitly the issue of the difference between 'definition' and 'properties' of a concept, as well as a concern for "essentiality" in a definition:

M.: whose area is "r^2"... improper English...

R.: *Would that be a definition of circle?*

M.: No, it's an element of the circle...

R.: *Why would you exclude [this definition] then? Can you show a figure which satisfies this definition and it is not a circle?*

(May suggests her spiral once again as an example of a figure with the same area of the circle, yet different from the circle)

R.: *Is there any other definition which you would like to eliminate?*

K.: *Yes, mine. (The reponse comes very quickly and forcefully)*

R.: *Why? (laughing)*

K.: *Because I just put anything I could possibly think of... And I was wrong: πr^2 is not the circumference, it's the area... I put 360 degrees but I didn't know... I was not able to put down a round answer, I just put what came to my mind...*

R.: *That's also what this definition F does. Why do you think we may not want to have a long list of properties?*

K.: *I am not saying that it would not be good, but...*

R.: *Oh, you would like to have put even more?*

K.: *I just did not remember...*

M.: But for a definition... it should be stated as simply as possible...

R.: *So, we want a definition to be able to identify only circles. And a long list of properties would probably do that even better. Then why you would not like it?*

M.: Why? Because a definition is something you have to remember... you don't want to remember all the little things... the whole list...

Both the issues of essentiality and the distinction between 'definition' and a set of properties of a given concept were too complex to be resolved completely at this point. We came back to these issues several times throughout the unit, especially in the third session, when we tried to use alternative definitions of 'circle' to solve problems such as "Find the circle passing through three given points" and "Find the measure of the interior angle of a regular pentagon inscribed in a circle".

Different kinds of concerns were raised by an analysis of def. E, which was not as familiar to the students as I had expected (despite the fact that they had recently completed a unit on graphing!). Perhaps because of their lack of background, the students were able to point out several important pieces of information which had been left out by the author of the definition - since such elements are usually taken for granted in the context of analytic geometry.

K.: *I don't know what x and y are...*

R.: *You are right, we have to say what x and y are, or it does not make any sense...*

K.: *Like, in mine, if I should do it over, I should say what r means...*

R.: *Let's say we were using graph paper...*

K.: *Oh, that makes sense!*

M.: But this is not the full sense of what a circle is... because you donot always have graph paper...

R.: *That's a good point. But we can say... if you have a circle, you can put on it graph paper...*

M.: With some work... This is a good definition, though, because it will only give circles...

R.: *But, how can you check if it does?*

This question was first answered by trying to plot some points satisfying the equation, and then by checking with the use of a compass whether they all belonged to a circle. However, later it also motivated a more rigorous deduction of the equation of the circle based on the equidistance property. Since the students had forgotten what had been done in previous courses, this provided the opportunity for genuine mathematical activity on their part, while they were creating some informal geometric arguments to convince each other of the essential equivalence of the metric and analytical definitions of 'circle'.

In the context of this exercise the occasion for another interesting digression occurred because of an observation by Kim regarding the similarity between the equation of a circle and the formula contained in the Pythagorean theorem.

K.: *But, how would you know... how could you say if that is the equation of the circle, but it is also the formula in the Pythagorean theorem for triangles...*

This confusion initiated a discussion on the difference between *variable* and *constant*, which we further developed in a later session when introducing the consideration of definitions of complex mathematical notions such as *variable* or *equation*.

The increased level of sophistication and criticism reached in the analysis of the definitions by the end of the session is evident in the following concerns expressed by May with regard to our revised versions of both definitions A and E:

M.: How would you figure out if something is a circle, if there is no measurement for the radius?

R.: *Ah! This is a good point!*

M.: What if they just say "circle", "draw a circle" and you are... what's its r?... So it doesn't work, because you don't know a circle if you don't know its radius.

R.: *Do you whink you need also to know where it is placed? Where is the center of the circle?*

M.: No...

This argument reveals an interesting expectation with respect to mathematical definitions; that is, that a definition should allow us to construct the object in question. However, in this case this expectation should be modified since *circle*, as most other mathematical concepts, is an abstraction - of all possible circles, with different radii and placed in different positions in the space. So its definition can at most allow us to construct an instance of the concept, once we have given a specific value to some of the general parameters left open in the definition itself - such as the value of r in the case of *circle*, for example. Unfortunately, the instructor did not seize the opportunity of expanding on this point at the time this happened - a regret felt more that once in this experience!

To conclude this section I would just like to report May's response when, at the end of the lesson, the instructor asked whether having two different yet acceptable definitions of 'circle' was disturbing:

M.: No, they are just two different ways of looking at it. That's a very healthy attitude! (laughs)

Episode 2: Creating a Definition of 'Polygon'

Defining *polygon* presented a real challenge to Kim and May, and brought to their attention some aspects and roles of mathematical definitions which they had not realized before.

From the very beginning, the students feld uneasy about the task of coming up with a good definition of *polygon*, because the notion was not clear in their mind. Even the activity of analyzing and refining some tentative definitions, which had been so successful in the case of *circle*, now proved rather frustrating. Our starting point in this case consisted of the definitions produced by the two students at the beginning of the unit:

Polygon: a geometric figure of straight lines that has no sides of equal length (May)
Polygon (quadrilater): sided geometric figure angles add up to be 360 (Kim).

Various figures were considered, such as:

FIGURE 3

Deciding whether each of these figures should be considered or not an example of *polygon* brought the students to refine May's original definition by adding the constraint that the figure should be closed (to avoid considering figure 3C as a possible polygon), and by eliminating the requirement that the sides should not be equal (so as to consider figures like the square as polygons). The refined definition now read:

Polygon: a closed figure of straight lines

However, in this activity it was difficult for the students to motivate their decisions, as shown in this brief exchange between the instructor and May:

R.: *Do we want this to be a polygon?*
M.: No, let's kick'm out. Let's be elite?!
R.: *What do we want to do with polygons?... so we can decide if certain kinds of figures will be, or not be.*
M.: I know we have several different kinds of polygons. It doesn't really have a definite shape... (mumbling).

The students welcomed with relief the instructor's suggestion to make the search for a definition of *polygon* more meaningful by attempting to prove some interesting results about polygons. To this end, the instrutor proposed the consideration of the following statement.

The sum of the interior angles of an n-sided polygon measures 180(n - 2) degrees.

Interestingly, this could be considered as a generalization of the property of quadrilaterals mentioned in Kim's definition.

R.: *How do you think we can prove something like this?*

M.: I don't know. Take a polygon as an example. (The instructor immediately draws one, a pentagon). We never really got to the definition of a polygon. *We think* this is a polygon.

R.: *Right, so for the moment we think it's a polygon. How to figure out the sum of the interior angles?*

M.: Put a circle around it that meets all the points.

R.: *(draws a very "skinny" polygon). What if the polygon is like this?*

K.: *(has a great insight). Make it into triangles.*

M.: Take a center point. But if it's really weird shaped you can't do it... Oh yes, you could do it.

R.: *(seizing another opportunity, but refusing to get side-tracked now). That's an interesting question: "When can we draw a circle around a polygon?" But keep it aside. (Getting back to the last issue raised by May). It seems that whenever I have a polygon I can pick a point in the center, more or less. Does it matter which point I pick? Maybe we should ask Kim why it is that you wanted to break it down into triangles like this?*

By developing Kim's insight, the students managed with some help from the instructor to find that the sum of the interior angles in this case was 3×180^{0}, thus confirming our tentative statement. The same procedure was then followed to derive the sum of the angles in an hexagon, confirming once again the statement, to the students' surprise:

M.: She's right.

K.: *I don't want you to be right.*

M.: It's too neat.

In the effort to find some counterexamples, May spontaneously suggested the consideration of a 5-pointed star:

FIGURE 4

She questioned, however, whether this kind of figure should be considered a polygon or not. Since we were now trying to prove a general statement about polygons, it was easier to decide this issue than it had been the case before. In the case of the 5-pointed star, in fact, one cannot even decide which angles are to be considered as interior angles, thus it would be impossible even to know how to interpret the statement we were trying to verify. We did not, therefore, want to consider the 5-pointed star as a polygon.

This discussion also helped us justify some of the previous decisions made about what figures should be considered or not as polygons (see previous figure 3), and at the same time resulted in a further refinement of our definition of polygon, which now read:

Polygon: a closed figure of straight lines which do not cross.

In order to deal somehow with the "star", however, May creatively suggested we consider only the outside lines:

FIGURE 5

With this modification, the star could then be considered a "normal" 10-sided polygon, and applying our procedure we could confirm once again the correctedness of the formula to compute the sum of the interiors angles - with great satisfaction from the part of the students.

The impacts of this experience on the students' conception of mathematical definition is best reflected in the students' responses to a review/questionnaire administered towards the end of the unit. Below I have reproduced the page regarding their work about the definition of 'polygon' - the students' written responses appear boldfaced.

> When trying to find a definition of 'polygon', we had some trouble to start with because we did not really know precisely what we wanted a 'polygon' to be - except that it was going to be a concept generalizing triangles, squares, pentagons, etc. So we started with a tentative definition, and "refined it" so that polygons would have some interesting properties.

We ended up with the following definition:

(i) A polygon is a closed geometric figure, with straight edges which do not cross.

Are you satisfied with this definition? How could we ever know if it is correct? Explain your answer.

M.: Not quite. Play w/ polygons. Test the theory. Use examples of figures that meet the requirements for the def. but don't meet properties. OR just T&E (NOTE: meaning "trial and error").

K: I am satisfied with this for the time being but I think possibly as I start to use them more and more, I may desire something more exact.

What are the main differences between this case, and the definition of concepts such as 'circle'?

M.: We already had a definition and a picture of the figure in our head before testing definitions and exceptions (with the circle).

K: Circle is just one figure, one concept
-a polygon is more general and includes many kinds of geometric figures (pictures of a triangle, square, parallelogram, and two more unusual concave polygons are given).

It is possible that in the future we may want to further modify the definition of 'polygon' or that of 'exponent'.

What is the value, then, of a preliminary, tentative definition?

M.: It's like a theory that hasn't been fully tested yet and therefore is open to modification.

K: Getting acquainted with a new concept - have something to base further defining on. You have to start with something.

Episode 3: Surprises with Circles in Taxigeometry

One of our last meetings was devoted to a discussion of the meaning of the usual definition of *circle* - "All the points in a plane equidistant from a given point" - in a new context, that of taxigeometry. I believe that this episode well illustrates the remarkable level of reflectivity, independence and critical stance assumed by the two students towards the end of the unit.

In preparation for this lesson, the students had first been introduced to the new situation by means of a story. The instructors' review of the different interpretation

given to the notion of "distance" in this situation (i.e., as the length of the minimal path along the streets of a city, or the lines of a grid), initiated an interesting discussion which reveals once again the critical attitude now assumed by our students, as well as their clear understanding of the notion of *distance*:

R.: *It seems reasonable to consider this (the shorter path along the lines of the grid) as the distance...*

M.: But if you were an airplane, then the distance would just be (and indicates a straight line connecting the two points)...

R.: *Right. But an airplane could not go through the buildings...*

M.: No... but an helicopter could!

R.: *(laughing) Okay! But we are human beings, we have to...*

K.: *(jokingly) But what if we are... "super hero"!*

M.: Yeah!

R.: *(laughing) That would give you a different distance...*

Once the constraints of the situation we wanted to work with had been clarified, the instructor asked the students to draw "all the points at distance 5 from a given point of the city".

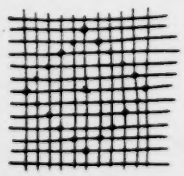
At these instructions, Kim immediately recognized the definition of circle:

K.: *May be a circle?*

R.: *Sounds like a circle, right? Let's try to construct it. Would it look something like this? (She draws a "usual" circle with radius 5).*

Meanwhile May had started figuring out her own version of the question. She had drawn all the points at distance 5 from the given point, using *taxidistance*, and commented:

FIGURE 6



M.: I think it should be a diamond.

This difference of opinion gave rise to a very lively and interesting discussion between the two students.

R.: *Now we have two theories.*

K.: *But ... you want all the points that are 5 away?*

R.: *Yes. Why is it not a circle?*

M.: Because you have to cut corners... There is no way to go up and down the buildings.

K.: *So what? She said "5 points away".*

To exploit this disagreement, I attempted to initiate an explicit discussion of the situation:

R.: *Can it be that both of you are right?*

M.: (without hesitation) Sure! It depends on what you want "distance" to be. It depends on whether you are on a graph or in a town. Uuh! It's fun. I like this! (She continues to embellish her graph paper).

R.: *Can we call them both circles? They both satisfy the definition!*

Since Kim was still in disagreement with the last statement, May explained to her once again why she thought that her "diamond" represented indeed all the points at the same *taxidistance* from the center point. However, at this point, May herself brought up a new concern. She noticed that even if her "diamond" satisfied the definition of circle we had agreed upon, it did not satisfy many of the other properties of circles. An excellent point, which shows once again May's increased understanding of the notion of definition.

M.: You could call that a circle, but if you went over it with properties, it wouldn't be a circle, because it's not 360 degrees, is it? (pointing at the interior angles of her diamond).

R.: *(instead, pointing at the angle at the center of the diamond). This is 360.*

M.: (disappointed) Oh, gee!

R.: *But, other properties may not work in this case. For example, the circumference is no longer $2\pi r$, or the area is not π^2. What others?*

These new considerations challenged May's initial acceptance of the diamond as a 'circle', and brought us once again to consider explicitly the nature of definitions and their relationship with the properties of an object. The search for a solution to this conflict brought May to an interesting compromise. She started to analyze the original

definition of circle in the attempt to identify either a vague term (which should be more precisely defined) or some misinterpretation on our part. Interestingly, her attention was not caught by the word "equidistant" (as it would seem most obvious) but by the interpretation of the condition of being "on a plane". This led her to the following argument, which is quite creative and logically sound (though some details may need refinement).

M.: I don't think so. We have *not* made a circle here. This is not... In the definition we said it was on a one-dimensional plane.

R.: *Isn't this a plane?*

M.: Yeah, but not "one-dimension" because of the buildings, this is more than one dimension.

R.: *The situation is not the same.*

M.: Yeah, I'm sure... In our definition we said "A geometric figure on a plane"... and this is *not* a plane... (becoming silly) this has chimneys... going way up. These are all like buildings that are coming up in 3D. So [the diamond] is *not* a circle. (Pointing to Raffaella's) *that's* a circle!

What a nice way to describe the Euclidean plane: a plane with no obstacles!

May had clearly grasped the notion that the context may modify the meaning of a definition, and had resolved her initial conflict by finding the "mistake". There did not seem to be, however, an acknowledgement that for the same reason the definition could be considered correct and would describe either figures depending on the context - a realization that would probably require a much more radical departure from the common view of the nature of mathematics.

Interestingly, this activity spontaneously suggested to May a desire to define precisely "diamond" - perhaps still in the effort to contradict the disturbing fact that her "diamond" and a circle could share the same definition. Both students got very involved in the task, and with minimal intervention from the part of the instructor, they proceeded along the same lines as initially done with the definition of 'circle' and that of 'polygon' - that is, they attempted to successively refine a tentative definition by considering many examples and counterexamples based on their intuitive notion of 'diamond'. In my opinion, this episode provides the best possible evidence of what the two students had internalized about the nature and role of definition, since they themselves posed the problem (that is, they felt the need for a definition of a concept which they had clear in their mind) and chose a reasonable procedure to approach a solution.

New Insights on Using Errors as Springboards for Inquiring Gained from the Teaching Experiment

As illustrated in the episodes just described, the effect of this unit on the students' understanding of the complex nature of mathematical definitions, and even more on their conceptions of and attitudes towards mathematics, was remarkable and went even beyond my initial expectations. These results were certainly due to many elements of the teaching experiment besides the use of errors as springboards - for example, the challenging nature of the topic addressed, the opportunity for flexibility and individual attention allowed by the small number of students, the attitude and role assumed by the instructor, or the personality of the two students involved. However, I also believe that the use of *errors as springboards* played an essential part in the success of our instructional unit.

First of all, the strategy of using *errors as springboard* was used consistently and often much more than once in every lesson. This shows that a use of *errors as springboards* was essential in the experience, and eliminating this aspect from the design of the unit would totally change its nature.

While I am open to the possibility that similar results in terms of understanding the nature of mathematical definitions and realizing the more humanistic aspects of mathematics could be reached through alternative methods, I would like to point out how a use of *errors as springboards* helped us achieve them in the teaching experiment:

- Errors definitely proved useful to make the students aware of some abstract issues regarding the nature of mathematical definitions, and to provide them with a concrete and accessible way to explore and discuss them. Clear evidence in this sense is provided, for example, by the activity of analyzing incorrect definitions of circle (which led the students to recognize the motivation and the importance of the requirements usually imposed on mathematical definitions).

- Errors often allowed the creation of situations of cognitive conflict, which in turn stimulated the students' inquiry and discussion. This happened especially when the students suggested alternative solutions, and instead of having the instructor tell them which one was correct they were forced to decide the issue on their own (think for example of the case of the interpretation of the definition of *circle in taxigeometry*). Often, the conflict and the inquiry which followed it brought the students to results which went beyond deciding which of the two alternative answers was the correct one - and this happened not only because of the instructor's intervention, but also as a result of the students' own

initiative (as again in *taxigeometry*, when May initiated a search for a definition of *diamond* in the effort to eliminate the confusion between the usual circle and taxicircles).

- Having to constantly deal with errors made the students more and more critical of their work, and less dependent on the instructor's authority to evaluate the results of their mathematical activities. Evidence of their increased critical stand can be found throughout the unit, but it is especially visible in their behavior when the instructor proposed to them (inadvertantly) an incorrect theorem in episode 2, when working with polygons.

- "Errors" which even mathematicians could not resolve (or, as May preferred to characterize them, "complications") generated considerable reflection and discussion. This was especially evident in Episode 3, as well as when the students encountered limitations in the extension of *exponent*, and seemed to have considerable impact on the students' conclusions about mathematics being "created by humans".

Finally, it may be interesting to look at the participants' perceptions about the strategy of *using errors as springboards for inquiry*. Because the use of errors as springboards was so embedded in the whole teaching approach and in the design of the unit, it was difficult to gather the students' explicit reactions to the strategy. The following excerpt from May's interview is especially illuminating in this regard:

L: *Uhm. Did you, did you, uhm, think that Raffaella used errors and the notion of error in mathematics while she was teaching this course to you?*

M: Oh, yeah that was the main point of it and invite creative thinking.

L: *In what ways do you think it was that?*

M: Uhm, we weren't really... before I said something about [it], not on this tape but before in class... I said something about how we were presented with errors. But we weren't really, we were presented with a question like: "How?" And then we would try to figure it out and we would come to errors first. Which is, you know, is a regular way of learning things, you make an error and you try to correct it and you just work through and, uhm, sorta, I guess that errors were a really big part of it. Just because that came with our learning process. But she didn't present them to us, she didn't give us something that was an error. Except on the final sheet after we had learned to see errors.

initiative (as again in *taxigeometry*, when May initiated a search for a definition of *diamond* in the effort to eliminate the confusion between the usual circle and taxicircles).

- Having to constantly deal with errors made the students more and more critical of their work, and less dependent on the instructor's authority to evaluate the results of their mathematical activities. Evidence of their increased critical stand can be found throughout the unit, but it is especially visible in their behavior when the instructor proposed to them (inadvertantly) an incorrect theorem in episode 2, when working with polygons.

- "Errors" which even mathematicians could not resolve (or, as May preferred to characterize them, "complications") generated considerable reflection and discussion. This was especially evident in Episode 3, as well as when the students encountered limitations in the extension of *exponent*, and seemed to have considerable impact on the students' conclusions about mathematics being "created by humans".

Finally, it may be interesting to look at the participants' perceptions about the strategy of *using errors as springboards for inquiry*. Because the use of errors as springboards was so embedded in the whole teaching approach and in the design of the unit, it was difficult to gather the students' explicit reactions to the strategy. The following excerpt from May's interview is especially illuminating in this regard:

L: *Uhm. Did you, did you, uhm, think that Raffaella used errors and the notion of error in mathematics while she was teaching this course to you?*

M: Oh, yeah that was the main point of it and invite creative thinking.

L: *In what ways do you think it was that?*

M: Uhm, we weren't really... before I said something about [it], not on this tape but before in class... I said something about how we were presented with errors. But we weren't really, we were presented with a question like: "How?" And then we would try to figure it out and we would come to errors first. Which is, you know, is a regular way of learning things, you make an error and you try to correct it and you just work through and, uhm, sorta, I guess that errors were a really big part of it. Just because that came with our learning process. But she didn't present them to us, she didn't give us something that was an error. Except on the final sheet after we had learned to see errors.

Consequently, when asked to react to the use made of errors in the unit, neither student could put in words the more radical effects of the strategy which had been observed from their behavior. Overall, they felt positive about using errors, and pointed out that they now looked at errors differently and thought they could learn from them:

M.: It helped, because, alot with me personally, because I feel like I can recognize my own errors alot easier now. It's like, before, when I would do a test and to these classes and then I would come to a point where I think I did it wrong, I really wouldn't know how to find out if I really did it right or not. And now I feel like I could do that easily. Which is a good thing.

[...]

L.: *Has this course given you a different approach to errors?*

K.: *Uhm, I guess I tried that a little bit, uhm, when I made an error in math class, in Judy's math class. I tried to look at my error and tried to figure out why I made that error and to see what that error led to. And to see if I could learn something from that, instead of just saying "Oh, I made this error, I did it wrong and this is the right way". To look at the reasons behind why I did that.*

From both the observation of the students' participation in the instructional sessions and their explicit comments in the interviews, one could notice a considerable difference in the two students' attitudes and reactions to the strategy employed. While both engaged in the learning and the discoveries resulting from the exercise, May was always more excited and enthusiastic, and seemed to take great satisfaction in challenging the given and in discovering things which "would not work". Kim, on the contrary, was more reticent, and sometimes seemed uncomfortable with our continuous challenge of what she had relied on so far. It seems worth discussing these different reactions while taking into account the students' prior experiences with mathematics. While Kim had been reasonably (even if unevenly) successful with "regular" math, and seemed to enjoy its systematicity and organization, May had resented her previous mathematics courses, refused to "memorize" when she thought she was expected to do so, and failed several course. It seems reasonable to suggest that sometimes the approach assumed in the unit could have made Kim unconfortable because it would take away some of her "security" and the source of previous success in mathematics - an important point which should be kept in mind in future implementations. On the contrary, May had almost nothing to lose in terms of mathematics success and esteem in her ability as a math student. Furthermore, some of the reasons for her past failure seemed to have to do with the very mechanistic view of mathematics which we were now trying to challenge and dispell. A use of *errors as springboards* thus found a ready subject in May, who was then able to exploit the potential of the strategy even beyond our expectations and hopes. I believe that there are many students, like May, whose success

in mathematics has been hindered by an unrealistic view of the discipline, and I hope that a use of errors as springboards could help them too in realizing the more humanistic aspects of mathematics and their own hidden mathematical potential.

V. Conclusions

The results both of the conceptual analysis conducted in sections II and III, and of the empirical experiences reported in section IV, has supported the initial claim that the educational potential of errors has not yet been fully exploited in mathematics instruction. By engaging the students themselves in the constructive use of mathematical errors - as *springboards for inquiry* - we can in fact provide our pupils with a powerful heuristic tool and with a wealth of opportunities for problem posing, problem solving and critical thinking activities within the context of mathematics instruction.

Further research is necessary to better evaluate the potential benefits which could be derived by employing the strategy of *using errors as springboards* in mathematics instruction, as well as to appreciate the many possible ways in which students could be engaged in a constructive use of errors so as to enhance their own construction of mathematical knowledge. The results presented in this paper are but the point of departure in recognizing the value and implications of using errors as *springboards for inquiry* in mathematics instruction.

At the same time, it is important to appreciate that the successful implementation of an *errors as springboards* approach will require the reexamination of other aspects of current mathematics instruction as well. As long as students perceive making errors as evidence of academic failure and the cause of lower grades, and continue to operate under time pressures to "cover" a rigidly prescribed curriculum sequence, it is not surprising that most of them react to errors with frustration and show only the desire "to be told how to get the right answer". It is also unlikely that students will be willing and interested in spending the considerable time and effort necessary to pursue an in-depth study of their errors, unless they expect such an activity to be sufficiently worthwile and rewarded.

To make students appreciate and take advantage of errors as *springboards for inquiry* is thus necessary that a compatible learning climate be established in the classroom. Among other things, this may require a change of emphasis from <u>product</u> to <u>process</u> in the evaluation of students' learning, the creation of incentives for conceptual

understanding and risk-taking, and some flexibility with respect to learning objectives so as to allow for the degree of leisure necessary to pursue the "digressions" which could be motivated by errors.

The challenge, however, seems worth meeting, since the experiences conducted so far suggest that using errors as springboards in mathematics instruction can be very rewarding in terms of increasing students' motivation and learning, engaging them in creative and open-ended mathematical activities, and making them more reflective as mathematics students.

This study has been partially supported by a grand from the National Science Foundation (# MDR-8651582). The opinions expressed in this paper, however, are solely the author's.

REFERENCES

Borasi, R. (1981). *Studio matematico e sperimentazione didattica sul concetto di distanza*. Tesi di Laurea (Unpublished Thesis). Universita' di Torino, Torino (Italy), Department of Mathematics.

Borasi, R. (1986a). *On the Educational Roles of Mathematical Errors: Beyond Diagnosis and Remediation*. Unpublished Doctoral Dissertation. State University of New York at Buffalo.

Borasi, R. (1986b). "Algebraic Explorations around the Error 16/64 - 1/4". *The Mathematics Teacher*, 79, 4, 246-248.

Borasi, R. (1987). "Exploring Mathematics Through the Analysis of Errors". *For the Learning of Mathematics*, 7, 3, 1-8.

Borasi, R. (1988). *Using Errors to Inquire into the Nature of Mathematical Definitions: A Teaching Experiment*. Preliminary report to the National Science Foundation (Grant # MDR-8651582).

Brown, S.I. and L. Callahan, eds. (1985). *Using Errors as Springboards for the Learning of Mathematics*. Special issue of *FOCUS: On Learning Problems in Mathematics*, 7, 3 and 4.

CIEAEM, (Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques) (1988). *The Role Errors Play in the Learning and Teaching of Mathematics*. Proceedings of the 39th Meeting of the Commission Internationale pour l'Etude et l'Amélioration de l'Enseignement des Mathématiques, Sherbrooke, ON (Canada). Les Editions de l'Université de Sherbrooke.

Kline, M. (1980). *Mathematics: The Loss of Certainty*. Oxford (UK): Oxford University Press.

Krause, E. (1986). *Taxicab-geometry*. New York, NY: Dover.

Kuhn, T. (1970). *The Structure of Scientific Revolutions*. Chicago, IL: The University of Chicago Press.

Lakatos, I. (1976). *Proofs and Refutations*. Cambridge (UK): Cambridge University Press.

Novak, J.D. and H. Helm, eds. (1983). *Proceedings of the International Seminar: "Misconceptions in Science and Mathematics"*. Cornell University, Ithaca, NY.

Paper, S. (1980). *Mindstorms: Children, Computers and Powerful Ideas*. Basic Books.

Papy, G. and F. Papy (1973). *Taximetrix*. Bruxelles (Belgium): Hachette.

Radatz, H. (1979). "Error Analysis in Mathematics Education". *Journal for Research in Mathematics Education*, 10, 3, 163-172.

Radatz, H. (1980). "Students' Errors in the Mathematical Learning Process". *For the Learning of Mathematics*, 1, 1, 16-20.

Vergnaud, G. (1982). "Cognitive and Developmental Psychology and Research in Mathematics Education: Some Theoretical and Methodological Issues". *For the Learning of Mathematics*, 3, 2, 31-41.

Des exemplaires du présent document sont en vente à l'adresse suivante:

Centre interdisciplinaire de recherche sur l'apprentissage
et le développement en éducation
Université du Québec à Montréal
Pavillon Sainte-Catherine Ouest
515, rue Sainte-Catherine Ouest
Local X-7115

Case postale 8888, succursale A
Montréal (Québec)
H3C 3P8

Tél.: (514) 282-6186

Pour commander, remplir le formulaire de réquisition
apparaissant à la dernière page du document.

Publié par le Centre interdisciplinaire de recherche sur l'apprentissage
et le développement en éducation

Conception graphique: Pierrette Précourt